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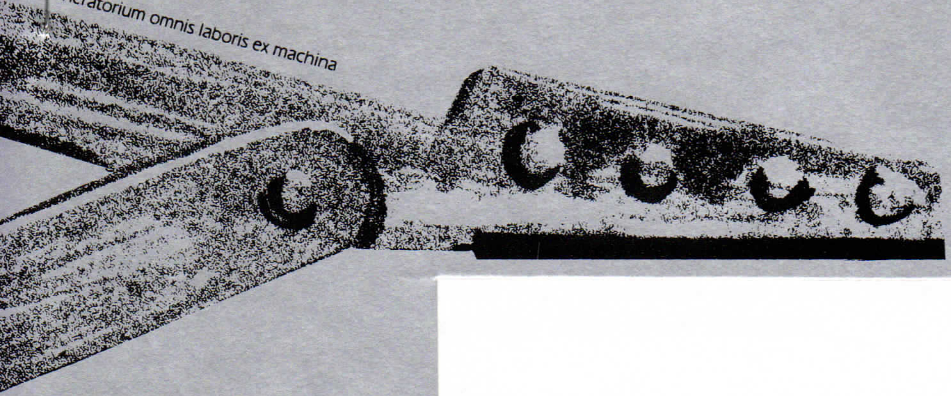

Representations based on Zero-crossings in Scale-Space

by

Robert A. Hummel

Technical Report No. 225
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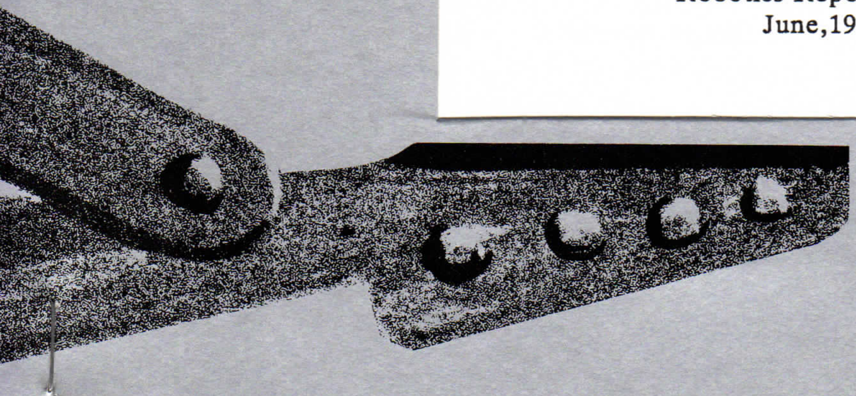

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Representations based on Zero-crossings in Scale-Space

Robert A. Hummel

Abstract

Using the Heat Equation to formulate the notion of scale-space filtering, we show that the evolution property of level-crossings in scale-space is equivalent to the maximum principle. We briefly discuss filtering over bounded domains. We then consider the completeness of the representation of data by zero-crossings, and observe that for polynomial data, the issue is solved by standard results in algebraic geometry. For more general data, we argue that gradient information along the zero-crossings is needed, and that although such information more than suffices, the representation is still not stable. We give a simple linear procedure for reconstruction of data from zero-crossings and gradient data along zero-crossings in both continuous and discrete scale-space domains.

1. Scale-space and zero-crossings

The use of multiresolution representations is an important idea for the analysis of signal and image data. Many data structures have been studied, including Gaussian pyramids, difference-of-Gaussian pyramids, Laplacian pyramids, and "scale-space" formulations [1, 2, 3]. The latter formulation, to be described briefly below, can be used as a continuous model of the other formulations. We will discuss the representation of data by zero-crossings in scale-space, and consider the stability of reconstruction methods.

The natural framework for the analysis of scale-space formulations of multiresolution representation is in terms of the heat equation [4, 5]. Specifically, let $f(x)$ be a bounded function defined for $x \in \mathbb{R}^n$. (Arbitrary dimensions can be handled with little additional fuss over the case $n = 1$; we will later comment on the case of a bounded domain $D \subseteq \mathbb{R}^n$). We define $u(x,t)$ to be a bounded solution to the heat equation:

$$\frac{\partial u}{\partial t} = \Delta u, \qquad \text{(Heat Equation)}$$

$$u(x,0) = f(x).$$

The solution is given by convolution against the fundamental solution to the Heat Equation, which for the domain $\mathbb{R}^n$ is given by

$$u(x,t) = \int_{\mathbb{R}^n} K(x-y,t) f(y) dy,$$

where

$$K(x,t) = (4\pi t)^{-n/2} e^{-|x|^2/4t}.$$

We see that $u(x,t)$ is obtained by blurring $f(x)$ by increasingly diffuse Gaussians, parameterized by $t>0$, with standard deviations σ satisfying $2\sigma^2 = 4t$. In computer vision, scale-space sometimes refers to the (x,σ) variables that can be used to reparameterize the domain of u. We retain the (x,t) parameterization to keep the linear Heat Equation relation for the function u.

Convolution by Gaussians is considered special for many reasons [6,4,7]. We see from the above analysis a relationship between Gaussian convolution, the Heat Equation, and the Laplacian operator. Of course, Gaussian convolution enjoys other properties; for example, the central limit theorem implies that Gaussian convolution is easy to implement by an iterative procedure. However, we also see the extent of the similarity of difference of Gaussians and the Laplacian of the Gaussian; namely, since $K(x,t)$ is itself a solution to the Heat Equation,

$$(\Delta K)\ (x,t) = \frac{\partial K}{\partial t}(x,t) = \lim_{\tau \to 0}(K(x,t+\tau) - K(x,t))/\tau.$$

That is, the difference of Gaussians is a good approximation to ΔK as the separation between the spread of the two Gaussians approaches zero (and the difference is scaled).

Filtering by the Laplacian of a Gaussian can be written in three ways:

$$\Delta K * f = K * \Delta f = \Delta(K * f).$$

If we denote the result by $v(x,t)$, we see that

(1) $v(x,t)$ is the $f(x)$ data filtered by the Laplacian of a Guassian.

(2) $v(x,t)$ is the solution to the Heat Equation with initial data Δf.

(3) $v(x,t)$ is $\Delta u(x,t)$, where u is the solution to the Heat Equation with initial data $f(x)$.

The zero set of $v(x,t)$ is the point set in (x,t) where $v = 0$. The set might be empty (for instance, if f is subharmonic or superharmonic; [8]) everything (if f is harmonic; [9]), or a proper subset of (x,t) space. In the latter case, zeros can be isolated points, lines, and surfaces (but never regions). We distinguish components of the zero set which form manifolds of codimension one:

Definition: The *zero-crossings* of $v(x,t)$ refers to the point set

$$\partial\{(x,t)\,|\,v(x,t) < 0\} \ \cap \ \partial\{(x,t)\,|\,v(x,t)>0\}. \ \blacksquare$$

Zero-crossings have been suggested for segmentation of imagery by edge detection [10], and for stereo matching and motion correspondence between pairs of images (e.g., [11]). It has also been suggested [7] that the zero-crossings are a nearly complete representation of Δf. Finally, Witkin [3] observes that zero-crossings in scale-space evolve as t increases, and are never created at some nonzero t . This property, discussed in [7] and in [6], ensures that zero-crossing surfaces are nested, one within another, enclosing regions containing the face $\{t = 0\}$, or forming a sheet meeting the face $\{t = 0\}$ and extending to $t = \infty$. The property can be given a precise statement:

Evolution property of zero-crossings: Let C be a connected component of the set of zero-crossings in the domain $\{(x,t)\,|\,x\in\mathbb{R}^n,\ T_1\le t\le T_2\}$, where $0\le T_1 < T_2$. Then $C\ \cap\ \{(x,t)\,|\,t = T_1\}\ne\emptyset$. ■

In the remainder of this paper, we wish to make two main points. First, we establish the equivalence of the "Evolution property for zero-crossings" and the classical maximum principle for parabolic partial differential equations, thereby allowing us to consider bounded domains and nonstationary convolutions. Second, we consider reconstructibility from zero-crossings. The principle result is that if knowledge of the location of the zero-crossings is supplemented with gradient information of v *at only those points in the zero-crossings*, then there is a simple scheme for reconstructing some of the data, but that even then numerical accuracy of the reconstruction is unstable.

2. The Maximum Principle

The classical maximum principle for the solution to the parabolic equation $\partial u/\partial t = \Delta u$ states (see, e.g., [12, 13, 14]):

Maximum Principle: Let $D \subseteq \mathbb{R}^n$ be open and bounded. Suppose u is a solution in $T = \{(x,t)\ |\ x\in D,\ 0<t<T\}$ of class C^2 which is continuous in the closure $\bar{T}$. Then u assumes its maximum at some point (x,t) for which either $x\in\partial D$ or $t = 0$. ■

Next, suppose that scale-space construction is denoted by the operator $v = Sg$, which is to say that the scaled function $v(x,t)$ is obtained from the initial data $g(x)$. In the previous section, we defined S to be $v(x,t) = K(\cdot,t) * g$, where $g(x) = \Delta f(x)$, but we can imagine more general operators. In any case, it is logical to make certain assumptions about S, although all we will require is that

(1) If $g(x)$ is continuous, then Sg is continuous.

(2) $S(-g) = -Sg$ for all g, and if $v = Sg$ and $\tilde{g}(x) = v(x,\tau)$, then $\tilde{v} = S\tilde{g}$, where $\tilde{v}(x,t) = v(x,t+\tau)$.

(3) If $g(x)\to 0$ as $|x|\to\infty$ and $v = Sg$, then for each t, $v(x,t)\to 0$ as $|x|\to\infty$.

We note that if S is defined by convolution with Gaussians as in Section 1, then the maximum principle holds for $v = Sg$ as long as g is continuous; further, conditions (1) - (3) hold.

Our first result is:

Proposition: The following are equivalent:

(i) The maximum principle holds for solutions $v = Sg$ using continuous initial data $g(x)$ satisfying $g(x)\to 0$ as $|x|\to\infty$.

(ii) The evolution property holds for level-crossings of solutions $v = Sg$ using a scale-space operator S satisfying (1) - (3) above and continuous $g(x)$ satisfying $g(x)\to 0$ as $|x|\to\infty$.

Proof: We first show that the maximum principle implies the evolution property. For if the evolution property fails for a level-crossing l for some $v(x,t)$, then by suitably transforming v, g, and l, we can assume that $l\ge 0$, and that there is a solution $v = Sg$, (with $g\to 0$ as $|x|\to\infty$) with a component C of $\{(x,t)\ |\ 0\le t\le T,\ v(x,t)>l\}$ disjoint from the plane $t = 0$. Let (x_0,t_0) be a relative maximum in C. Then since C is open, there is a bounded cylinder in C with

(x_0, t_0) in the interior, which is in violation of the maximum principle.

Conversely, if the maximum principle fails, then for some cylinder $D \times [0,T]$ and some $v(x,t)$ given by $v = Sg$ with $g(x) \to 0$ as $|x| \to \infty$, the maximum of v occurs either in the interior or the top of the cylinder. Either way, there is a value l less than the maximum but greater than the values on the bottom and sides of the cylinder. Thus there is a component of the level-l crossing within the wedge $0 \le t \le T$ which lies entirely within the interior of the cylinder, and thus does not meet $\{t = 0\}$. So the evolution property for level-crossings is violated. ■

We illustrate the utility of the proposition with three observations. First, given the equivalence with the maximum principle, any proof of the evolution property for zero-crossings that does not either use the maximum principle or essentially redo the proof of the maximum principle is highly suspect. Since the maximum principle is slightly delicate, especially in the absence of strong regularity assumptions, the former course seems more appropriate.

Second, using the version of the gradient Hopf maximum principle for the Heat Equation [13], it is not hard to show that knowledge of the zero-crossings together with gradient information along with the zero-crossings (or even just one zero-crossing contour) is sufficient to determine $v(x,t)$ uniquely [15]. In section 3, we give a more constructive discussion of this point; but it is interesting that the maximum principle establishes this uniqueness.

Finally, we see that any scaling method obeying the maximum principle will yield the evolution property for zero-crossings. Under fairly severe restrictions, this leads one to Gaussian convolution [6], but more general scaling methods are possible. For example, blurring by a parabolic operator of the form $\partial u / \partial t = Lu$, where L is a uniformly elliptic linear second order differential operator with non-constant coefficients will certainly still give a maximum principle. In fact, L can be nonlinear [16]. Moreover, suppose we replace $\mathbb{R}^n$ with a bounded domain $D \subseteq \mathbb{R}^n$, and insist on data $f(x)$ with compact support in the interor of D. We may then define $v = Sg$, where $g = \Delta f$, by solving

$$\partial v / \partial t = \Delta v \quad \text{in } D \times (0,\infty),$$

$$v(x,0) = g(x) \quad \text{for } x \in D$$

$$v(x,t) = 0 \quad \text{for } x \in \partial D .$$

This scaling is not given by convolution against a Gaussian, since the domain is bounded, but nonetheless obeys a maximum principle, and gives the same evolution property.

3. Completeness

We return to a consideration of zero-crossings of data $v(x,t)$ obtained by filtering initial data $f(x)$ defined for $x \in \mathbb{R}^n$ by the Laplacian of Gaussians, $\Delta K(\cdot,t)$. The question we wish to address is: to what extent do the zero-crossings represent $f(x)$? Clearly, $f(x)$ can at best be reconstructed to within an arbitrary additive harmonic function and a scalar multiple. However, if we assume that $f(x) \to 0$ as $|x| \to \infty$, then only the multiplicative constant is of concern.

Yuille and Poggio [17] make the observation that if $g(x)$ and hence $f(x)$ is polynomial in x, and if $n=1$, then reconstruction from zero-crossings is theoretically possible. They also refer to the validity of the observation for larger n.

We note, however, that when $g(x)$ is a polynomial in $x \in \mathbb{R}^n$ for any n, then $v(x,t)$ is a polynomial in $(x,t) \in \mathbb{R}^{n+1}$. Accordingly, the zero-crossings are part of the *analytic varieties* of the polynomial v as studied in algebraic geometry. It is well known that the varieties in C^n determine the complex polynomial defined on n complex variables. It is not as commonly used, but nonetheless true, that an n-dimensional subportion of the intersection of the analytic variety with $\mathbb{R}^{n+1}$ also determines the polynomial [18]. Thus the case of polynomial data can be settled with algebraic geometry.

However, since the determination of a polynomial by its varieties is essentially an analytic continuation result, stability of the reconstruction is unlikely. That is, small errors in measurement of the zero-crossings could lead to arbitrarily large errors in the determination of $g(x)$. Put differently, there can be widely different initial data leading to nearly identical zero-crossing data.

Worse, settling the case for polynomial data says little about the case of continuous initial data. Although the Stone-Weierstrass theorem says that a continuous function can be uniformly approximated by a polynomial on a compact set, the zero-crossings depend on the initial data globally, and the dependence can't be localized. Further, the lack of stability means that the approximation is irrelevant. The situation is similar to the fact that a polynomial of a single variable with all real roots is determined by its zeros, but that given all the zeros of a continuous function, one knows nothing more than the zeros.

In fact, there are known examples of pairs of functions $f_1(x)$ and $f_2(x)$ such that the corresponding $v_1(x,t)$ and $v_2(x,t)$ have identical zero-crossings at all levels of resolution. John Daugman supplies the example (for two space dimensions) of $f_1(x_1,x_2) = \sin x_1$, and $f_2(x_1,x_2) = (\sin x_1)(2 + \cos x_2)$.

However, if the zero-crossing data is supplemented with knowledge of the gradient data at the zero-crossings, then reconstruction of at least some of the data $g(x)$ is theoretically possible by a quite easy procedure, given below. Details of these ideas were reported earlier in an unpublished work [15]. The use of gradient data for the representation also appears in [4], but the gradient data there is not limited to the zero-crossings. The use of gradient data along zero-crossings is discussed in [19]. Many researchers have noted from a casual observation of zero-crossings of image data that zero-crossings with large gradient magnitudes are of greater significance than those with low gradient magnitudes.

3.1. Continuous Case

Specifically, let Ω be a *bounded* connected component of $\{(x,t) \mid t \geq 0,\ v(x,t) \neq 0\}$, and denote by D the set $\{x \in \mathbb{R}^n \mid (x,0) \in \Omega\}$, and by Γ the zero-crossing $\partial\Omega \cap \{t>0\}$. Let τ be a value such that $\tau > \sup\{t \mid (x,t) \in \Omega\}$. Next, we set $\tilde{g}(x) = g(x)$ for $x \in D$, and $\tilde{g}(x) = 0$ elsewhere. Finally, let $b(x)$ be the $\tilde{g}(x)$ data blurred to the level τ:

$$b(y) = \int_{\mathbb{R}^n} K(y-x,\tau)\tilde{g}(x)dx.$$

Using Green's theorem, it is easy to show:

Proposition:

$$b(y) = \int_{\Gamma} K(y-x,\tau-t)\nabla v(x,t)\cdot\mathbf{n}d\sigma,$$

where $\mathbf{n}$ is a surface normal to Γ at (x,t), and $d\sigma$ is surface area measure. ■

Thus given the zero-crossing Γ and $\nabla v(x,t)$ for $(x,t)\in\Gamma$, then the blurred data $b(x)$ can be constructed by a simple linear process. The original data $\tilde{g}(x)$ can be reconstructed by deblurring the $b(x)$ data [20]. Deblurring is, of course, a classic unstable process. The situation is not hopeless, however, since $\tilde{g}(x)$ has known compact support, which might be used to advantage, and also since errors that occur are predominantly in high frequency components, which might not be as essential to visual interpretability.

The lesson of this section, ultimately, is that even for bounded zero-crossings supplemented with gradient data along the zero-crossing, reconstruction is still unstable. We defer a remark on relaxing the constraint that the zero-crossing be bounded until the next subsection, where we consider a discrete version of the result of this section.

3.2. Discrete Data

For simplicity, we treat the case of one unbounded space dimension, although the results extend easily. We are given data f_i, $i = \cdots,-1,0,1,\cdots$, and define

$$g_i = \frac{1}{4}f_{i-1} - \frac{1}{2}f_i + \frac{1}{4}f_{i+1}.$$

We define the filtered data $v_{i,k}$ recursively:

$$v_{i,0} = g_i,$$

$$v_{i,k+1} = \frac{1}{4}v_{i-1,k} + \frac{1}{2}v_{i,k} + \frac{1}{4}v_{i+1,k}.$$

We also define the blurring kernel

$$K_{i,k} = \frac{1}{4^k}\binom{2k}{i+k}.$$

Both v and K satisfy a discrete version of the Heat Equation, namely

$$u_{i,k+1} - u_{i,k} = \frac{1}{4}u_{i-1,k} - \frac{1}{2}u_{i,k} + \frac{1}{4}u_{i+1,k}.$$

It is not hard to prove a discrete analogue of the evolution property for zero-crossings. The key, as one might suspect from Section 1, is a discrete version of the maximum principle, which is easy to establish.

Let Ω be a bounded 4-connected collection of pixels (i,k) with a nonempty set $\mathcal{D} = \{i \mid (i,0)\in\Omega\}$. Let T be an upper bound $T > \max\{k \mid (i,k)\in\Omega\}$, and define b_i to be the data g_i, $i\in\mathcal{D}$, blurred to level T:

$$b_i = \sum_{j \in D} K_{i-j,T} \cdot g_j .$$

Finally, let

$$\partial_{(\pm 1,0)}\Omega = \{(i,k) \in \Omega \,|\, (i \pm 1, k) \notin \Omega\},$$

$$\partial_{(0,1)}\Omega = \{(i,k) \in \Omega \,|\, (i, k+1) \notin \Omega\},$$

$$\partial_{(0,-1)}\Omega = \{(i,k) \in \Omega \,|\; k>0,\ (i, k-1) \notin \Omega\}.$$

Then simple but messy algebra allows us to show
Proposition:

$$4b_j = \sum_{\epsilon=-1,1} \; \sum_{(i,k)\in\partial_{(\epsilon,0)}\Omega} \left[\frac{v_{i,k}+v_{i+\epsilon,k}}{2} [K_{i+\epsilon-j,T-k-1} - K_{i-j,T-k-1}] - \frac{K_{i+\epsilon-j,T-k-1}+K_{i-j,T-k-1}}{2} [v_{i+\epsilon,k} - v_{i,k}] \right]$$

$$+ \sum_{(i,k)\in\partial_{(0,1)}\Omega} 4v_{i,k+1} \cdot K_{i-j,T-k-1}$$

$$- \sum_{(i,k)\in\partial_{(0,-1)}\Omega} 4v_{i,k} \cdot K_{i-j,T-k} .$$

To reconstruct data by the above equation, choose a connected component of $\{(i,k) \,|\, v(i,k)>0\}$, (or respectively <0). If the component extends to infinity in either coordinate, truncate the domain to become a convenient bounded collection of pixels, and denote the result by Ω. We store the sets $\partial_{(\pm 1,0)}\Omega$, $\partial_{(0,\pm 1)}\Omega$, and D as defined earlier. For pixels (i,k) in $\partial_{(1,0)}\Omega$ (respectively $\partial_{(-1,0)}\Omega$), we store the information $v_{i,k}$ and $v_{i+1,k}$, (respectively $v_{i,k}$ and $v_{i-1,k}$). For pixels (i,k) in $\partial_{(0,1)}\Omega$, we store the data $v_{i,k+1}$ and for $\partial_{(0,-1)}\Omega$ pixels we store $v_{i,k}$. Using the above equation, we choose a T and reconstruct the blurred data b_j. To reconstruct the data g_i for $i \in D$, it suffices to deblur the b_i data by solving for g_i in the linear equations defining b_i. In fact, the system is overdetermined, although still poorly conditioned, especially if $|D|$ or T is large.

In order to make the computations feasible, it is necessary to modify the formulas for a bounded spatial domain. We in fact solved a bounded domain problem, with $-N \le i \le N$, setting $v_{i,k} = 0$ for $i = \pm N$. The blurring kernel K is changed by this modification, but the proposition carries over with little change.

In Figure 1, we show a 1-D signal g_i, and the zero crossing separating positive and negative regions of the associated $v_{i,k}$. Applying the above procedure to the central positive component Ω, we obtain the reconstructed b_i data shown in Figure 2. The true b_i data is identical to essentially machine precision. Finally, using the method of pseudoinverses to deblur the data shown in Figure 2, we obtain g_i for i in the middle range, as shown in Figure 3. This is to be compared with the true initial data in Figure 1a. The poor correspondence is due to the fact that the deblurring problem is poorly conditioned; better deblurring results are obtained if the amount of deblurring is very small. However, this requires that the top of zero-crossing contour enclosing the data occurs after not too many blurring steps. The lesson learned here is that although reconstruction is in theory possible, practical reconstruction may be impossible.

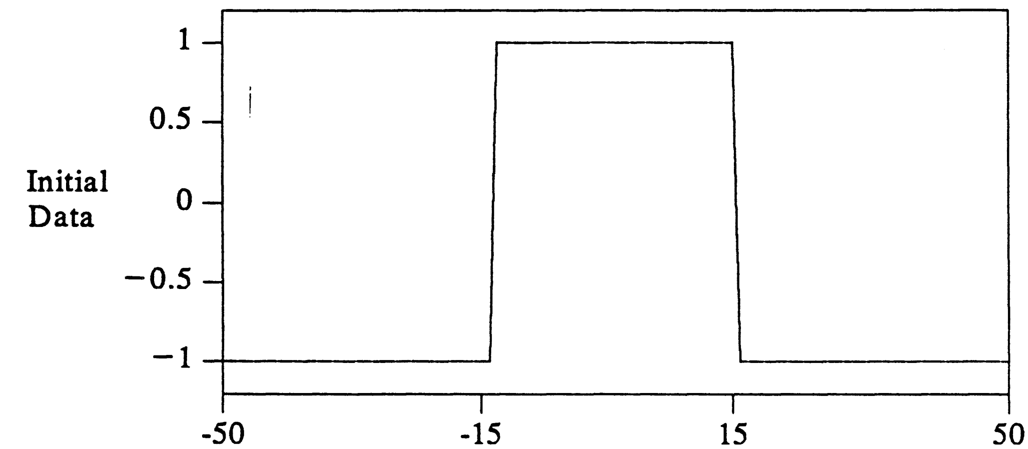

Figure 1a. Initial g(i)

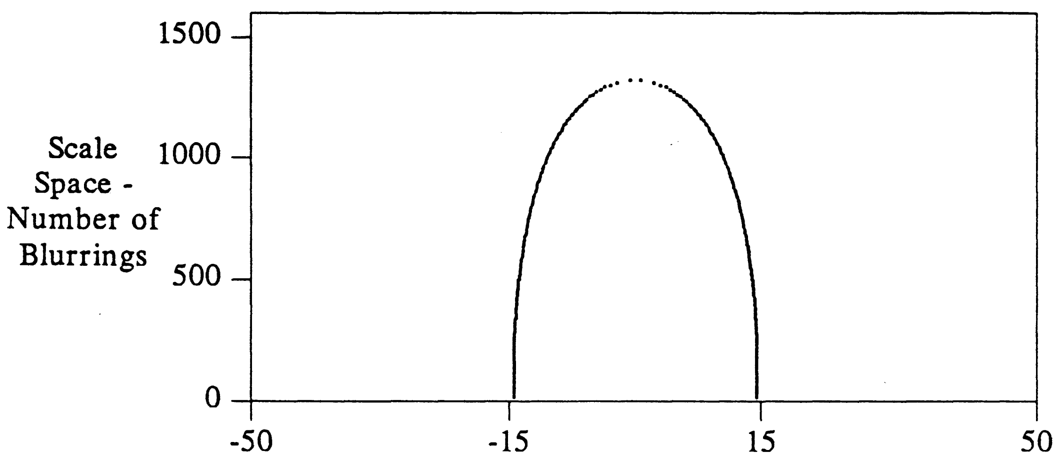

Figure 1b. Zero-crossings

Figure 1. An initial function g_i and the zero crossing pixels in $v_{i,k}$, where $v_{i,k}$ blurs the initial data by scaling in k.

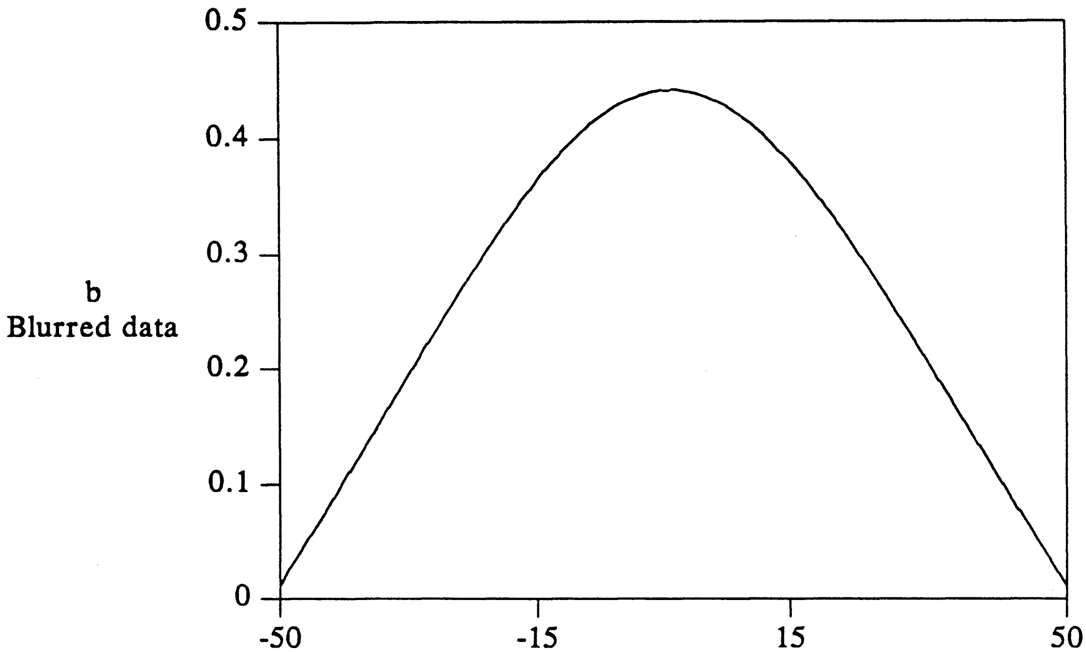

Figure 2. The reconstructed function b_i using the formula from Proposition 2. The data represents the result of blurring the g_i data restricted to the central positive interval, zero extended elsewhere, to a level k above the top of the zero crossing for $v_{i,k}$ shown in Figure 1 above. The reconstruction uses only information about $v_{i,k}$ along the zero-crossing, and is nearly exact to machine precision.

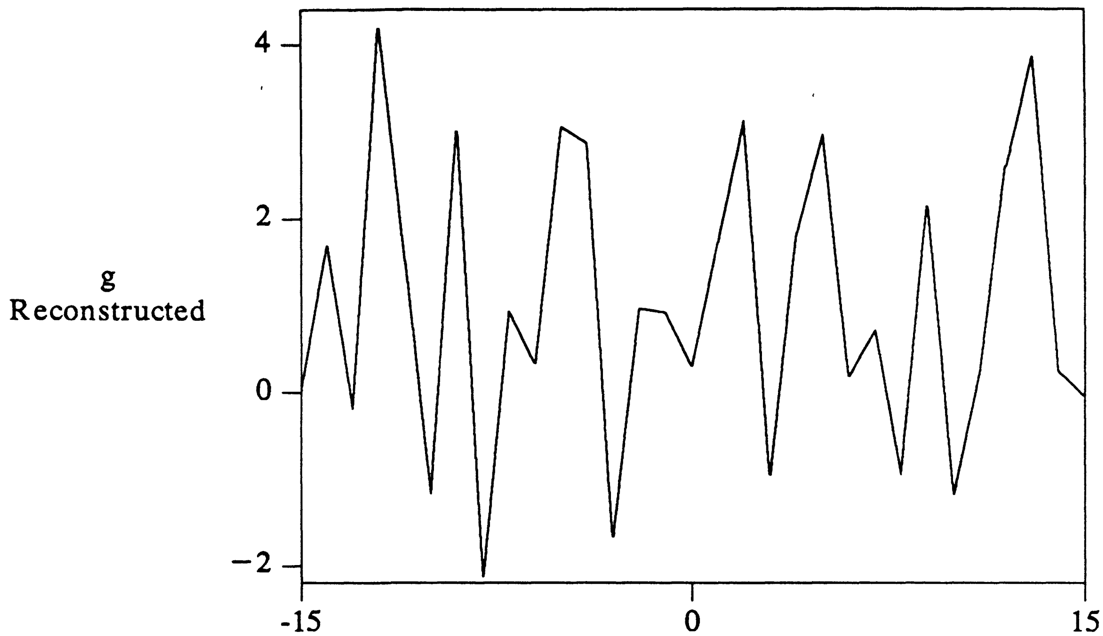

Figure 3. An attempt at reconstructing g_i using the data b_i from Figure 2. The attempt fails because the matrix equation $K\mathbf{g} = \mathbf{b}$ relating the **g** data to the **b** data is poorly conditioned, even though there are many more **b** values than **g** values. Thus the small errors due to round-off in representing the b_i data are magnified when reconstructing g_i. The singular value decomposition software in Creve Moler's "matlab" package was used.

4. Comments

Zero-crossings of scale-space filtered data seems like an unlikely form of representation of data. The results presented here suggest that even when supplemented with gradient data along the zero-crossings, the representation is still unlikely. However, the instability of the representation does not completely deny its utility, since it might happen that the classes of functions mapping to similar representations share properties essential for interpretation. A required step in the validation of the utility of a representation is an analysis of the invariant properties of signals that yield similar representations. A necessary condition is that attempted reconstructions differ from originals in unessential ways, (from the standpoint of interpretation). The methods outlined here should prove useful in verifying or disproving this necessary aspect of establishing a viable representation.

Acknowledgements

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