Gaussian Blur and the Heat Equation: Forward and Inverse Solutions [†]

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Abstract

Gaussian blur, or convolution against a Gaussian kernel, is one of the most common models for image and signal degradation. We are concerned with the inverse of this process, or Gaussian de-blurring. As in the process of blurring, we seek a linear deblurring kernel. Although the inverse of a Gaussian cannot be represented exactly as a convolution kernel in the spatial domain, by restricting the space of allowable functions to polynomials of fixed finite degree then a convolution inverse does exit. Constructive formulas for the de-blurring kernels are derived in terms of Hermite polynomials. For image polynomials of fixed degree N, the corresponding kernel gives stable deblurring among the class of functions which are Gaussian filtered versions of data well approximated by polynomials of degree N and less. Stated differently, the de-blurring kernels are *pseudo-inverses* of the Gaussian convolution operator.

1. Introduction

Given an image or a signal, the realization of any practical system for processing it must introduce some amount of degradation. Since almost all of these systems consist of several stages, each of which contributes to the degradation, they often compose into what appears to be a Gaussian degradation. In this paper we shall be concerned with inverting this process, or the de-blurring of Gaussian blur.

Our model of blur is as a spatially invariant Gaussian point spread function within a linear system. Formally this leads to convolutions, as follows. Let $f(x)$ denote the original image function, $x \in \mathbb{R}^n$. Then the observable - but blurred - function $h(x)$ is given by:

$$h(x) = K(x,t) * f(x) = \int_{\mathbb{R}^n} K(x-\xi,t) f(\xi) d\xi,$$

where

$$K(x,t) = \frac{1}{(4\pi t)^{n/2}} e^{-|x|^2/4t}$$

is the Gaussian kernel, whose extent is parameterized by $t > 0$. It is normalized to have unit mass.

The problem of Gaussian de-blurring can now be formulated: How can the original data $f(x)$ be reconstructed when *only* $h(x)$ and the amount of blurring t are known? Again, we shall formulate this as a convolution, and we seek a filter $D(x,t)$ such that

$$f(x) = D(x,t) * h(x) = D(x,t) * K(x,t) * f(x)$$

for $f(x)$ among a class of functions.

Our motivation for choosing this problem is two-fold. Firstly, many practical imaging configurations are structured in a manner that introduces blur either optically or for other reasons (e.g. computerized tomography [Herman, 1980]), and the Gaussian is the natural first approximation to this blur. And sensors are becoming far more reliable at higher light levels, leaving deterministic sources of blur more salient. Techniques for reducing this blur are thus of practical importance. There are even applications in physiological optics, such as the de-focusing that automatically takes place for objects outside of the depth of field of an accomodated eye.

Our second motivation is theoretical. It is well known that while the deblurring problem is in general non-invertible from Fourier considerations and unstable, [†] it is nonetheless possible to achieve acceptable deblurring under certain conditions. One way to accomplish this is by means of a pseudo-inverse [‡] which is an exact inverse under restricted conditions. Although such results have been available in the mathematical literature for some time [John, 1955], they are not widely known within the computational vision and image processing communities. Rather, the image processing community typically formulates the problem purely in discrete terms by applying algebraic pseudo-inverse techniques [e.g., Pratt, 1978]. But this obscures the analytical structure of the process, leaving central notions such as the *order* of the de-blurring pseudoinverse implicit. Pseudo-inverses imply notions of approximation, and one would like a formulation in which the degree of this approximation could be made explicit. Then one could understand how the structure of the de-blurring kernels changed as a function of the order of approximation.

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For a more detailed version of this paper see [Hummel, Kimia, Zucker].

† One must be clear about the fundamental distinction between a "stable" or "unstable" problem (in the Numerical Analysis literature it is usually refered to as a well- or ill-conditioned problem [e.g. see Stewart, 1973]) as opposed to a stable or unstable algorithm for a given problem. Henceforth, stability will mean differently when applied to a problem or an algorithm.

‡ Also, refered to as a "generalized inverse."

In this paper, we derive kernels which can be used to deblur a fixed amount of Gaussian blur. They accomplish this inverse process *exactly*, and stably. among polynomials of fixed degree. Our analysis uses Hermite polynomials, a natural choice for reasons that will become clear shortly. The explicit formulas for the de-blurring filters are given in the main theorem in Section 5. Since the analysis leading to this theorem is technical. we provide motivating background in the next section.

2. Background

2.1 Blurring and Diffusion

There is a fundamental connection between blurring, deblurring. and the heat equation. It is provided by the structure of the Gaussian distribution. as the following example illustrates. Consider a rod of infinite length onto which an impulse of heat is placed at some position. As time evolves. the heat will diffuse and the original impulse will spead out. By basic physics the resulting temperature distribution will approximate a Gaussian whose extent depends on the elapsed time (see e.g., Feynman, 1963). By superposition, the model for the temperature distribution along the rod at any time is the initial temperature distribution convolved with a Gaussian. This is the physically realized solution to the heat equation †. The spatial parameter for the Gaussian depends on how much time has evolved. and the diffusion process effectively *blurs* the initial temperature distribution incrementally. In the notation introduced in Sec. 1. if $f(x)$ is the initial temperature distribution. then $h(x,t) = K(x,t) * f(x)$ is the blurred distribution after t units of time. Formally, this is an initial value problem. and can be stated as follows: given $f(x)$ and t, find $h(x,t)$ satisfying

$$\Delta h = \partial h/\partial t, \qquad h(x,0) = f(x).$$

We, of course. will interpret $f(x)$ as an image.

Two basic observations follow from this formulation of the blurring problem that will be important in the analysis. First, note that the space of initial distributions that can be blurred is a large one; it essentially corresponds to any function for which the convolution integral is defined, and clearly includes some discontinuous ones. Second, suppose that a function $f(x)$ has been blurred for some time, say t_0, resulting in $h(x,t_0)$. This resultant function could subsequently be blurred further, say to t_1, with $t_1 > t_0$. These two blurring operators, each of which may have its own physical justification, results in one composite Gaussian operator. Indeed, by the central limit theorem, other blurring operators compose into approximate Gaussians when iterated.

2.2 Deblurring and the Inverse Heat Problem

Since deblurring is the inverse of blurring, the preceeding connection between blurring and diffusion suggests that deblurring can be modeled as a diffusion running backwards in time. Blurring is the forward problem. and deblurring is the inverse problem. Formally, the problem of reconstructing $f(x)$ given $h(x)$ and t is the inverse heat equation problem, since the function $h(x)$ represents a distribution of heat after t units of time, where $f(x)$ is the initial $t = 0$ distribution.

As in the forward or blurring problem. which was modeled as convolutions of the original data against a "blurring kernel" (a Gaussian), our goal now is to find "deblurring kernels", or kernels against which the blurred data can be convolved to yield the deblurred original. However, the mathematics is not straightforward. There are a number of technical differences which make the deblurring problem more difficult than blurring. While the blurring (or heat diffusion) problem can be solved for almost all distributions (i.e., the solution is just a smoothed version of the initial data), the inverse problem is defined only for a restricted class of functions. Running "time" backwards makes it impossible, in general, to reconstruct the original data $f(x)$ from the blurred data $h(x)$. First. not all functions h(x) are blurred versions of some original function $f(x)$. Secondly, the blurring operator is not a one-to-one mapping in a general function space. There exist pairs of distinct functions. $f(x)$ and $\hat{f}(x)$, which yield the same blurred function $h(x)$. Finally. in a general function space the deblurring problem is horribly ill-conditioned. In other words, arbitrary small perturbations in the given function $h(x)$ can lead to large changes in the reconstruction of $f(x)$.

These difficulties are so severe that one might be pessimistic about any progress toward discovering deblurring kernels. However, the deblurring problem can be given a *pseudo-inverse* formulation. which leads to a well-conditioned problem. We formulate the pseudo-inverse problem for polynomial data in section 3, and present the deblurring kernels for polynomials in (5.5). The structure of these kernels is a function of the order of approximation. revealing how the solution to the problem changes as the data become more complex.

3. Pseudoinverse Formulation

Let $\mathcal{T}$ denote the blurring opertor. which takes functions in a large normed space Λ into much smoother functions, also in Λ. Although $\mathcal{T}$ is a continuous operator, for nearly any choice of Λ, $\mathcal{T}$ has no continuous inverse defined on its range.

The idea of a pseuoinverse is as follows: Consider a closed subspace $\mathcal{M} \subseteq \Lambda$. The image of $\mathcal{M}$ under $\mathcal{T}$ will also be a closed subspace, and so if Λ is a complete Hilbert space, one can pose the problem

$$\text{Given} \quad h \in \Lambda, \qquad \text{find} \quad f \in \mathcal{M} \qquad \text{minimizing} \quad \|\mathcal{T} f - h\|.$$

The solution f to this minimization problem is the pseudo-inverse of h under the map $\mathcal{T}$ on Λ relative to the subspace $\mathcal{M}$, and will be denoted $\Lambda_{\mathcal{M}}^{-1} h$.

In our case, we set $\Lambda = \mathcal{L}^2(e^{-x^2}dx)$, an enormous Hilbert space. which contains distributions which are not tempered. We set $\mathcal{M} = P_N$. the space of polynomials of degree N and

† the so-called "source kernel" Widder, 1975.

less. In section 4, we will note that M is $\mathcal{T}$-invariant. Since $\mathcal{M}$ is finite dimensional and $\mathcal{T}$ is one-to-one, $\mathcal{T}$ is an isomorphism of $\mathcal{M}$ onto $\mathcal{M}$. Thus the problem of finding the pseudo-inverse of h is equivalent to finding f such that $\mathcal{T}f$ is the orthogonal projection of h onto $\mathcal{M}$. An algorithm for computing f can therefore be constructed by projecting h to h' on $\mathcal{M}$, and then solving the finite dimensional problem $\mathcal{T}f = h'$. Clearly, this process is stable for fixed N.

In section 5, we present the solution to the deblurring problem on $\mathcal{M}$, so that the problem $\mathcal{T}f = h'$ is solved by a convolution

$$f = D_N * h'.$$

It is evident that for $h \in \Lambda$, and h' the orthogonal projection of h onto $\mathcal{M} = P_N$,

$$D_N * h = D_N * h'.$$

Thus the entire algorithm, projection onto $\mathcal{M}$ and inverting $\mathcal{T}$ on $\mathcal{M}$, can be represented by a single convolution. In fact, the kernels D_N given in section 5 are unique in having this double property.

4. The Deblurring Problem

Consider the operator Ω_t defined on $\mathcal{L}^2(\mathbb{R})$ by the equation

$$(\Omega_t f)(y) = \int_{-\infty}^{\infty} \frac{1}{2\sqrt{\pi t}} e^{-x^2/4t} f(y-x)dx.$$

For $t > 0$, Ω_t is a compact symmetric bounded linear operator on $\mathcal{L}^2(\mathbb{R})$ mapping into $\mathcal{L}^2(\mathbb{R})$. This operator has many special properties, such as

$$\Omega_t \circ \Omega_s = \Omega_{t+s}$$

and

$$u(x,t) = (\Omega_t f)(x).$$

It satisfies the *heat equation*

$$\Delta u = u_t,$$

with

$$u(x,0) = f(x);$$

see [Bers, John, Schechter]. If we denote the Fourier transform of a function g(x) by $\hat{g}(\omega)$, then Ω_t is a multiplier operator given by

$$(\widehat{\Omega_t f})(\omega) = e^{-\omega^2 t}\hat{f}(\omega)$$

By means of this formula, Ω_t can be extended to operate on the class of temperate distributions S' of Fourier transformable distributions [Hörmander, 1983]. In particular, $\Omega_t f$ is defined for any polynomial f.

We will specialize to the case of $t = \frac{1}{4}$, and set

$$\mathcal{T}f = \Omega_{1/4}f = \frac{1}{\sqrt{\pi}}e^{-x^2} * f. \qquad (4.1)$$

By suitably scaling the spatial parameter $x \in \mathbb{R}$. Ω_t, $t > 0$, can be seen to be equivanlent to $\mathcal{T}$ operating on a rescaled version of f.

From the Fourier multiplier formula

$$(\widehat{\mathcal{T}f})(\omega) = e^{-\omega^2/4}\hat{f}(\omega). \qquad (4.2)$$

and the fact that $e^{-\omega^2/4} \neq 0$ for all ω, it is clear that $\mathcal{T}$ is one-to-one on any space of Fourier transformable functions. Further, since the inverse of the multiplier, $e^{\omega^2/4}$, has no inverse Fourier transform, the inverse of $\mathcal{T}$ is not representable as a convolution, nor can be applied to the general space of all Fourier transformable functions. Instead, we can restrict the domain of $\mathcal{T}$, and then represent its inverse as a convolution on the range of $\mathcal{T}$. Many such restricted domains are possible. In the next section, we consider $\mathcal{T}$ restricted to the class of polynomials of degree N or less.

5. Polynomial Domains

Let P_N denote the space of polynomials over $\mathbb{R}$ of degree less than or equal to N. The monomials $\{1, x, x^2, \ldots, x^N\}$ form a basis for P_N. If this basis is orthonormalized with respect to the inner product

$$(f,g) = \int_{-\infty}^{\infty} f(x)g(x)e^{-x^2}dx, \qquad (5.1)$$

then the basis of Hermite polynomials $\{H_0, H_1, \ldots, H_N\}$ result. The Hermites can be represented explicitly:

$$H_n(x) = n!\sum_{m=0}^{[n/2]}(-1)^m \frac{(2x)^{n-2m}}{m!(n-2m)!}. \qquad (5.2)$$

or by the Rodrigues formula:

$$H_n(x) = (-1)^n e^{x^2}\frac{d^n}{dx^n}(e^{-x^2}), \qquad (5.3)$$

see, e.g., [Courant and Hilbert, 1962] or [Lebedev, 1965].

Our main result is that $\mathcal{T}^{-1}$ restricted to P_N can be represented by a convolution with an explicit kernel $D_N(x)$:

Theorem: For $f \in P_N$ and $g = \mathcal{T}f$, then

$$f = D_N * g \qquad (5.4)$$

where

$$D_N(x) = e^{-x^2}\sum_{k=0}^{[N/2]}\frac{(-1)^k}{\sqrt{\pi}k!2^k}H_{2k}(x). \qquad (5.5)$$

The detailed proof of the above theorem, together with the mathematics that leads to it can be found in [Kimia, Zucker] and [Hummel, Kimia, Zucker] and is omitted here for brevity.

It is interesting to compare the form of $D_N(x)$ with standard enhancement filters. For example, for $N = 3$,

$$D_3(x) = \frac{2}{\sqrt{\pi}}e^{-x^2}(1-x^2),$$

which is a not uncommon high emphasis filter (see, e.g., the papers by E. Mach in [Ratliff, 1965] and Rosenfeld and Kak, 1976].

6. Higher Dimensions

The Gaussian blur operator is given by

$$\mathcal{T} f(x) = \int_{\mathbb{R}^n} \pi^{-n/2} e^{-(x-y)^2} f(y) dy. \tag{6.1}$$

Due to the separability of the kernel and Fubini's theorem. $\mathcal{T}$ can be decomposed into n iterated blurrings:

$$\mathcal{T} = \mathcal{T}_1 \circ \mathcal{T}_2 \circ \dots \circ \mathcal{T}_n \tag{6.2}$$

$$(\mathcal{T}_i f)(x) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} e^{-(x_i - y_i)^2} f(x_1, \dots, y_i, \dots, x_n) dy_i \tag{6.3}$$

It can be shown, [Hummel, Kimia, Zucker], that deblurring of blured polynomials of degree N can be accomplished by convolution against the kernel

$$D_N{}^*(x) = D_N(x_1) D_N(x_2) \dots D_N(x_n). \tag{6.4}$$

Thus the situation in higher dimensions is similar to the one dimensional case. The deblurring convolution kernel is separable, and will be of the form $e^{-x^2} P(x)$, where $P(x)$ is a polynomial of degree nN in $x \in \mathbb{R}^n$. Figure 1 shows a plot of $D_N{}^*$ for $n = 2, N = 3$.

7. Conclusions

Gaussian blur is one of the most common forms of degradation affecting signals and images. It is unfortunately non-invertible in general, but pseudo-inverses are possible. In this paper we formulated a precise version of the Gaussian deblurring problem, and obtained formulae for the kernels of deblurring filters in terms of Hermite polynomials. One then simply needs to convolve these kernels against (blurred) images to effect deblurring. As the order of the kernel increases, the space on which deblurring is exact increases as well.

The mathematics used in formulating the deblurring kernels were based on the heat equation. The connection between blurring and the heat equation is provided by the Gaussian: the spread of any heat distribution is governed by convolutions against a Gaussian kernel. Deblurring then amounts to solving the heat equation backwards in time.

However, backward solutions to the heat equation are notoriously unstable. Nevertheless, we have been able to show that stable deblurring is possible in principle for a class of image functions, and, perhaps more importantly, that some degree of stable deblurring is possible in practice for real images. The example in the paper was obtained using the most straightforward implementation. More serious attention to numerical issues, such as arithmetic precision and quadrature, could possibly lead to even better results.

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FIGURES

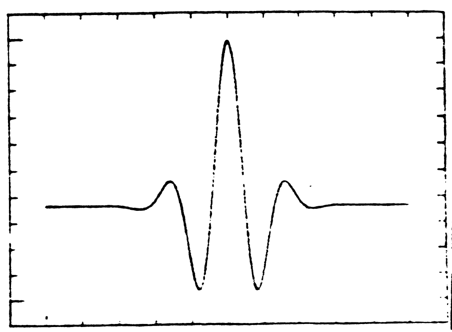

Figure 1. One dimensional deblurring kernel, $D_N, N = 9$.

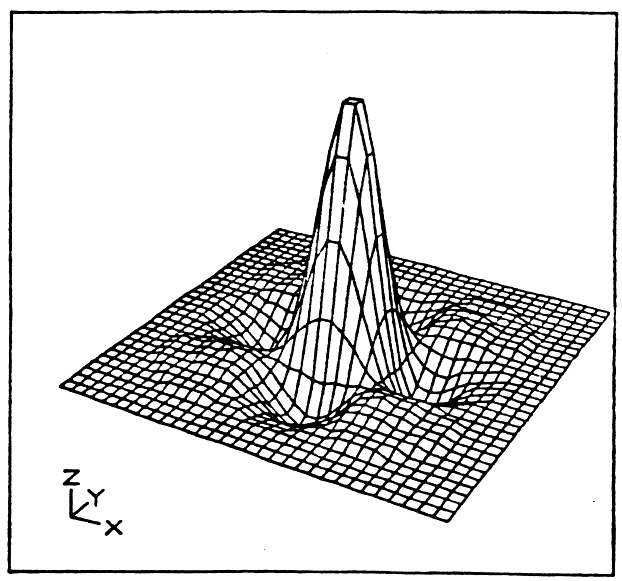

Figure 2. Two dimensional deblurring kernel, $D_N(x_1, x_2)$, $N = 3$. Note the sign changes in the kernel surrounding the central positive peak.